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# Permutation-type solutions to the Yang–Baxter and other $n$ -simplex equations

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**Abstract.** We study permutation-type solutions to  $n$ -simplex equations, that is, solutions whose matrix form can be written as  $R_{i_1 \dots i_n}^{j_1 \dots j_n} = \prod_{\alpha=1}^n \delta_{A_{\alpha} i_{\beta} + B_{\alpha}}^{j_{\alpha}}$  with some  $n \times n$  matrix  $A$  and vector  $B$ , both over  $\mathbb{Z}_D$ . With this ansatz the  $D^{n(n+1)}$  equations of the  $n$ -simplex equation reduce to a  $[\frac{1}{2}n(n+1)+1] \times [\frac{1}{2}n(n+1)+1]$  matrix equation over  $\mathbb{Z}_D$ . We have completely analysed the 2-, 3- and 4-simplex equations in the generic  $D$  case. The solutions show interesting patterns that seem to continue to still higher simplex equations.

## 1. Introduction

The Yang–Baxter equation (YBE, or 2-simplex equation) is the fundamental equation of solvable models in  $(1+1)$  dimensions. For lattice models it guarantees the commutativity of the transfer matrix, and for particle scattering it implies solvability through the factorization of the scattering matrix [1, 2]. Therefore, in order to construct interesting solvable models one needs interesting solutions. For this reason the YBE has been studied extensively and indeed many solutions are known [1, 3], especially in the two-state case [4].

When one tries to generalize these solvable models to  $(2+1)$  dimensions, either by considering three-dimensional lattices or the scattering of straight strings, one obtains Zamolodchikov’s tetrahedron equation (3-simplex equation) as the fundamental equation [2, 6], whose solutions are needed for further development. Unfortunately only a few solutions are known for this equation [5–9] and when one proceeds to still higher dimensions and to the corresponding higher simplex equations very little is known.

The difficulties associated with these equations come mainly from sheer numbers, the  $D$ -state  $n$ -simplex equation is actually a set of  $D^{n(n+1)}$  equations on  $D^{2n}$  variables (in the nonconstant case  $(n+1)D^{2n}$  variables). Because of this, one is forced to make a rather restrictive ansatz in order to obtain any solutions at all. One method is to take some definite high-level structure (Lie algebra, chiral Potts) coming from somewhere else and apply it to the present situation. Our approach is complementary to this, the ansatz given below is defined in rather simple terms and we will then determine *all* solutions within this class.

Let us recall the standard set-up for the  $n$ -simplex equations. As usual we assume that we have linear operators  $\mathcal{R}$  which for the  $n$ -simplex case are assumed to act on a product of  $n$  identical vector spaces  $V$ , i.e.  $\mathcal{R} : V^{\otimes n} \rightarrow V^{\otimes n}$ . Let  $e_i$  be the  $D$  basis vectors of  $V$ . Since we want to perform algebra with the indices of the basis vectors it would be nice if the indexing formed a finite field. If  $D$  was a prime this would be possible with  $\mathbb{Z}_D$ ,

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integers modulo  $D$ , which is what we consider in this paper. However, some aspects of the following derivation work even if the indices just form a ring, for example with  $\mathbb{Z}_4$ .

To the operator  $\mathcal{R}$  we associate a numerical matrix with  $n$  pairs of indices by

$$\mathcal{R}(e_{i_1} \otimes \dots \otimes e_{i_n}) = R_{i_1 \dots i_n}^{j_1 \dots j_n} (e_{j_1} \otimes \dots \otimes e_{j_n}). \tag{1}$$

(Here, and elsewhere, in this paper, summation over repeated indices is assumed.) The  $n$ -simplex equation itself is defined on  $V^{\otimes [n(n+1)/2]}$ , and the linear operators operate trivially in all but the  $n$  spaces indicated by the subscripts, e.g.  $\mathcal{R}_{12}(e_{i_1} \otimes e_{i_2} \otimes e_{i_3}) = R_{i_1 i_2}^{j_1 j_2} (e_{j_1} \otimes e_{j_2} \otimes e_{i_3})$ , or in the general case with  $K_\alpha \in \{1, \dots, N\}$ ,  $N = \frac{1}{2}n(n+1)$ ,

$$(R_{K_1 \dots K_n})_{i_1 \dots i_N}^{j_1 \dots j_N} = R_{i_{K_1} \dots i_{K_n}}^{j_{K_1} \dots j_{K_n}} \prod_{\substack{k=1 \\ k \neq K_\alpha, \forall \alpha}}^N \delta_{i_k}^{j_k}. \tag{2}$$

In this paper we consider the first few constant simplex equations, those given by the 2-simplex or vertex Yang–Baxter equation

$$\mathcal{R}_{12} \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{R}_{23} \mathcal{R}_{13} \mathcal{R}_{12} \tag{3}$$

the 3-simplex or tetrahedron equation

$$\mathcal{R}_{123} \mathcal{R}_{145} \mathcal{R}_{246} \mathcal{R}_{356} = \mathcal{R}_{356} \mathcal{R}_{246} \mathcal{R}_{145} \mathcal{R}_{123} \tag{4}$$

and the 4-simplex equation

$$\mathcal{R}_{1234} \mathcal{R}_{1567} \mathcal{R}_{2589} \mathcal{R}_{3680} \mathcal{R}_{4790} = \mathcal{R}_{4790} \mathcal{R}_{3680} \mathcal{R}_{2589} \mathcal{R}_{1567} \mathcal{R}_{1234}. \tag{5}$$

In terms of the multi-indexed matrices defined in (2) the above operator equations imply, respectively,

$$R_{j_2 j_3}^{k_2 k_3} R_{j_1 k_3}^{k_1 l_3} R_{k_1 k_2}^{l_1 l_2} = R_{j_1 j_2}^{k_1 k_2} R_{k_1 j_3}^{l_1 k_3} R_{k_2 k_3}^{l_2 l_3} \tag{6}$$

$$R_{j_3 j_5 j_6}^{k_3 k_5 k_6} R_{j_2 j_4 k_6}^{k_2 k_4 l_6} R_{j_1 k_4 k_5}^{k_1 l_4 l_5} R_{k_1 k_2 k_3}^{l_1 l_2 l_3} = R_{j_1 j_2 j_3}^{k_1 k_2 k_3} R_{k_1 j_4 j_5}^{l_1 k_4 k_5} R_{k_2 k_4 j_6}^{l_2 l_4 k_6} R_{k_3 k_5 k_6}^{l_3 l_5 l_6} \tag{7}$$

$$R_{j_4 j_7 j_9 j_0}^{k_4 k_7 k_9 k_0} R_{j_3 j_6 k_8 l_0}^{k_3 k_6 k_8 l_0} R_{j_2 j_5 k_8 k_9}^{k_2 k_5 l_8 l_9} R_{j_1 k_5 k_6 k_7}^{k_1 l_5 l_6 l_7} R_{k_1 k_2 k_3 k_4}^{l_1 l_2 l_3 l_4} = R_{j_1 j_2 j_3 j_4}^{k_1 k_2 k_3 k_4} R_{k_1 j_5 j_6 j_7}^{l_1 k_5 k_6 k_7} R_{k_2 k_5 j_8 j_9}^{l_2 l_5 k_8 k_9} R_{k_3 k_6 k_8 j_0}^{l_3 l_6 l_8 k_0} R_{k_4 k_7 k_9 k_0}^{l_4 l_7 l_9 l_0}. \tag{8}$$

In addition to the above, some other equations have appeared in literature, for example the Frenkel–Moore equation [10]. For a general formulation of the various types of equations, see [11].

## 2. Formulation with the permutation ansatz

In this paper we consider only permutation-type operators, that is those which transform one product of basis vectors into another simple product. In matrix form this means that there is precisely one nonzero ( $= 1$ ) entry in each column and row. For the  $D$ -state  $n$ -simplex equation there are  $(D^n)!$  different matrices to consider, and a brute force check of them is out of the question except for  $D = 2, n = 2$  which contains 24 permutation matrices (the next cases  $(2^3)!$  = 40 320 and  $(3^2)!$  = 362 880 might still be possible). We will therefore make the further assumption that the dependence between the basis vectors is *linear*, that is,

$$\mathcal{R}(e_{i_1} \otimes \dots \otimes e_{i_n}) = e_{A_1^\alpha i_\alpha + B_1} \otimes \dots \otimes e_{A_n^\alpha i_\alpha + B_n} \tag{9}$$

(where the summation over  $\alpha$  runs from 1 to  $n$ ) for some nonsingular  $n \times n$  matrix  $A$  and  $n$ -vector  $B$ , both having entries from  $\mathbb{Z}_D$ . In terms of the  $R$ -matrix this means that

$$R_{i_1 \dots i_n}^{j_1 \dots j_n} = \delta_{A_1^\alpha i_\alpha + B_1}^{j_1} \dots \delta_{A_n^\alpha i_\alpha + B_n}^{j_n} \equiv \delta(A, B). \tag{10}$$

The main advantage of this ansatz is that the problem of solving the  $D$ -state  $n$ -simplex equation can be reduced to handling ordinary matrices over  $\mathbb{Z}_D$ , as will be shown below. This simplifies the problem considerably. Furthermore, although possible applications normally imply further conditions on the solutions, permutation matrices are such fundamental objects that there is a good chance they are acceptable in most cases, and we believe that the ansatz is not an unnatural starting point.

In order to write the  $n$ -simplex equations in terms of  $A$  and  $B$  let us further define (in analogue with (2))

$$(A_{K_1 \dots K_n})_i^j = \begin{cases} A_\alpha^\beta & \text{if } i = K_\alpha, j = K_\beta \text{ for some } \alpha, \beta \\ \delta_i^j & \text{otherwise} \end{cases} \tag{11}$$

$$(B_{K_1 \dots K_n})_i = \begin{cases} B_i & \text{if } i = K_\alpha \text{ for some } \alpha \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

so that

$$(R_{K_1 \dots K_n})_{i_1 \dots i_N}^{j_1 \dots j_N} = \prod_{\mu=1}^N \delta_{(A_{K_1 \dots K_n})_\mu^{i_\nu} + (B_{K_1 \dots K_n})_\mu}^{j_\nu} \tag{13}$$

where now the  $\nu$  summation runs from 1 to  $N$ .

In the homogeneous case, that is with  $B \equiv 0$ , the above correspondence between  $R$  and  $A$  means that the  $n$ -simplex equation with ansatz (10) becomes an  $N \times N$  matrix equation over  $\mathbb{Z}_D$ . For example, the 2-simplex equation becomes

$$(A_{12})_i^k (A_{13})_k^m (A_{23})_m^l = (A_{23})_i^k (A_{13})_k^m (A_{12})_m^l \tag{14}$$

where  $A_{KL}$  are  $3 \times 3$  matrices with entries from  $\mathbb{Z}_D$  as given in (11) (for the explicit form see (23)).

In the nonhomogeneous case with  $B \neq 0$  a matrix formulation can also be obtained if we add a fictitious index space 0 and write

$$R_{i_1 \dots i_n}^{j_1 \dots j_n} = \delta_{A_1^{i_\alpha} + B_1 i_0}^{j_1} \cdots \delta_{A_n^{i_\alpha} + B_n i_0}^{j_n} \delta_{i_0}^{j_0} = \prod_{\mu=0}^n \delta_{\tilde{A}_\mu^{i_\nu}}^{j_\nu} \tag{15}$$

When this is immersed in the larger spaces we write the new index as the last one and then for the  $n$ -simplex case we obtain the  $[\frac{1}{2}n(n+1)+1] \times [\frac{1}{2}n(n+1)+1]$  matrix

$$\tilde{A}_{K_1 \dots K_n} = \begin{bmatrix} A_{K_1 \dots K_n} & B_{K_1 \dots K_n} \\ 0 & 1 \end{bmatrix} \tag{16}$$

(We use square brackets when writing out these index matrices.) For example the 2-simplex equation becomes

$$\begin{bmatrix} A_{12} & B_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{13} & B_{13} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{23} & B_{23} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{23} & B_{23} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{13} & B_{13} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_{12} & B_{12} \\ 0 & 1 \end{bmatrix} \tag{17}$$

and expanding this yields (14) for  $A$  and

$$A_{12}A_{13}B_{23} + A_{12}B_{13} + B_{12} = A_{23}A_{13}B_{12} + A_{23}B_{13} + B_{23} \tag{18}$$

for  $B$ . The higher simplex equations have equally simple matrix form. In fact, formally the equations now look exactly as in (3)–(5) with  $\tilde{A}$  instead of  $R$ , but the interpretation is different: for  $\tilde{A}$  we have ordinary matrix products.

### 3. Symmetries

Before starting to solve the equations it is necessary to discuss their symmetries. For one thing, we only want to list the basic solutions from which the others are obtained by the allowed transformations. It is well known [8] that the  $n$ -simplex equations are form-invariant under discrete transformations of index transposition and index reversal. Now we should also see if these transformations preserve the linear permutation structure and what they imply on  $A$  and  $B$ .

#### 3.1. Index transposition of $R$

If  $R_{i_1 \dots i_n}^{j_1 \dots j_n}$  is a solution of the  $N$ -simplex equation, then  $(IR)_{i_1 \dots i_n}^{j_1 \dots j_n} := R_{j_1 \dots j_n}^{i_1 \dots i_n}$  is also a solution. This is easy to see from the structure of the equation.

Let us now see what the above symmetry implies for the index matrix  $A$ . From the definitions above it follows that

$$(IR)_{i_1 \dots i_n}^{j_1 \dots j_n} = \prod_{\alpha=1}^n \delta_{A_{\alpha}^{\beta} j_{\beta} + B_{\alpha}}^{i_{\alpha}} = \prod_{\alpha=1}^n \delta_{(IA)_{\alpha}^{\beta} i_{\beta} + (IB)_{\alpha}}^{j_{\alpha}} \quad (19)$$

and by comparing the two expressions we find that if  $R = \delta(A, B)$  is a solution, then  $(IR) = \delta(A^{-1}, -A^{-1}B)$  is a solution, that is,  $(IA) = A^{-1}$ ,  $(IB) = -A^{-1}B$ .

It is easy to see that this is also an invariance of the  $\tilde{A}$  equation. As a matrix equation it is clearly invariant under matrix inversion (which furthermore does not change the location of the inserted pieces of the unit matrix) and  $\begin{bmatrix} A & B \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}B \\ 0 & 1 \end{bmatrix}$ .

#### 3.2. Index reversal of $R$

It is also easy to see that if  $R_{i_1 \dots i_n}^{j_1 \dots j_n}$  is a solution then  $(CR)_{i_1 \dots i_n}^{j_1 \dots j_n} := R_{i_n \dots i_1}^{j_n \dots j_1}$  is a solution. We have

$$(CR)_{i_1 \dots i_n}^{j_1 \dots j_n} = \prod_{\alpha=1}^n \delta_{A_{n+1-\alpha}^{\beta} i_{\beta} + B_{n+1-\alpha}}^{j_{\alpha}} = \prod_{\alpha=1}^n \delta_{(CA)_{\alpha}^{\beta} i_{\beta} + (CB)_{\alpha}}^{j_{\alpha}} \quad (20)$$

and the comparison yields  $(CA)_{\alpha}^{\beta} = A_{n+1-\alpha}^{n+1-\beta}$ ,  $(CB)_{\alpha} = B_{n+1-\alpha}$ , that is, reflection across the centre of the matrix or vector.

Since the  $\tilde{A}$  equation is a matrix equation it is invariant under any permutation of the set over which the summation is taken: if  $\tilde{A}_{\alpha}^{\beta}$  solves the equation, then  $(\sigma \tilde{A})_{\alpha}^{\beta} := \tilde{A}_{\sigma(\alpha)}^{\sigma(\beta)}$  where  $\sigma$  is any permutation operator, is also a solution. However, we have to keep intact the structure of inserted parts of the unit matrix in the various terms, and then it appears that only the above reversal is possible.

#### 3.3. Transposition of $A$

From the point of view of  $A$  the matrix equations have one more discrete symmetry: they are invariant also under transposition. However, this does not seem to correspond to any obvious invariance of  $R$ . In the following it will turn out that often the transposition of a solution  $A$  is also obtained by the central reflection (accompanied with parameter changes). However, this is not always true, and when it is not, it turns out that often the accompanying  $B$  will also be different.

This is a rather interesting result from the point of view of studying the structure of the equations. Normally, imposing an ansatz on the solutions restricts the symmetries,

because the symmetries of the equation may not be symmetries of the ansatz. In the present case this happens with the continuous transformation below. However, the opposite can also happen: in the present case the ansatz leads to a new formulation which has its own obvious symmetries, and some of these do not seem to have any counterpart at the original level.

### 3.4. Gauge transformations

The gauge transformation  $R_{K_1 \dots K_n} \rightarrow (QR)_{K_1 \dots K_n} = Q_{K_1}^{-1} \dots Q_{K_n}^{-1} R_{K_1 \dots K_n} Q_{K_1} \dots Q_{K_n}$ , is also an invariance of the  $n$ -simplex equations. Now that  $R$  is made out of delta-functions the transformation matrix  $Q$  must also be of that form, i.e.

$$Q_i^j = \delta_{ui+v}^j \quad (Q^{-1})_j^k = \delta_{u^{-1}j-u^{-1}v}^k \quad u, v \in \mathbb{Z}_D.$$

(If  $D$  is not prime,  $u^{-1}$  is not always defined.) A simple calculation shows that if  $R = \delta(A, B)$  is a solution, then  $QR := \delta(A, QB)$ , where

$$(QB)_\alpha := uB_\alpha + (1 - \sum_\gamma A_\gamma^\nu)v \tag{21}$$

is also a solution. Thus only  $B$  can change, and we can in fact put one  $B_\beta = 0$ , if  $\sum_\gamma A_\beta^\gamma \neq 1$ . Later we will find that for many solutions the inhomogeneous  $B$  part is such that it can be completely eliminated by this gauge transformation.

In order to understand this as an invariance of equations (14) and (18) we note first that (14) can be written as

$$\begin{aligned} A_{12}A_{13}(1 - A_{23}) + A_{12}(1 - A_{13}) + (1 - A_{12}) \\ = A_{23}A_{13}(1 - A_{12}) + A_{23}(1 - A_{13}) + (1 - A_{23}). \end{aligned}$$

If we now sum over the rightmost index of this equation and take its linear combination with equation (18) we obtain (21) for  $(QB)$ .

Matrix equations are invariant under a much larger group of similarity transformations:  $A \rightarrow O^{-1}AO$ , but now that we have to preserve the structure of having inserted pieces of the unit matrix into  $A_{K_1 \dots K_n}$  these similarity transformation are allowed only with the matrix

$$O = \begin{bmatrix} x\mathbb{I} & y\mathbb{I} \\ 0 & 1 \end{bmatrix} \text{ corresponding to the above.}$$

## 4. Results for the 2-simplex equation

The details for the Yang–Baxter or 2-simplex case are as follows. In the homogeneous case we write

$$R_{i_1 i_2}^{j_1 j_2} = \delta_{a i_1 + b i_2}^{j_1} \delta_{c i_1 + d i_2}^{j_2} \tag{22}$$

so that  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  and then the 2-simplex equation becomes

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{bmatrix} \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix} \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

(In [12] similar matrices, but with the entries also being matrices, are used to define a dynamical system.) This yields five equations,

$$\begin{aligned} abc = 0 \quad bcd = 0 \quad bc(b - c) = 0 \\ b(ad + b - 1) = 0 \quad c(ad + c - 1) = 0 \end{aligned} \tag{24}$$

whose solutions are discussed below.

4.1.  $D$  is prime

Recall that we are working with integers modulo  $D$ , the number of states. If  $D$  is prime there are no divisors of zero, and we can solve the equations with conventional rules of algebra. It is easy to show that in this case there are precisely four nonsingular solutions:

$$\begin{aligned} A_2^{(1)} &= \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} & A_2^{(2)} &= \begin{bmatrix} a & 1-ad \\ 0 & d \end{bmatrix} \\ A_2^{(2r)} &= \begin{bmatrix} a & 0 \\ 1-ad & d \end{bmatrix} & A_2^{(3)} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \quad (25)$$

Whether these are really different depends on  $D$ . Note, for example, that for  $D = 2$  cases (2) and (2r) reduce to case (1); the same holds with  $a = d = 2$  when  $D = 3$ . Solution (2r) is obtained from (2) by central reflection, and there is actually no need to mention it separately.

When the inhomogeneous part  $B = [x, y]^t$  is included we have to solve (18), which amounts to

$$b(x + ay) = 0 \quad c(y + dx) = 0 \quad x(c + d - bc - 1) = y(a + b - bc - 1). \quad (26)$$

The solutions then split further and we obtain

$$\begin{aligned} [A|B]_2^{(1a)} &= \left[ \begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \end{array} \right] & [A|B]_2^{(1b)} &= \left[ \begin{array}{cc|c} a & 0 & (a-1)z \\ 0 & d & (d-1)z \end{array} \right] \\ [A|B]_2^{(2)} &= \left[ \begin{array}{cc|c} a & 1-ad & -az \\ 0 & d & z \end{array} \right] & [A|B]_2^{(3)} &= \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right]. \end{aligned}$$

However, we have not yet used the gauge freedom. For  $[A|B]_2^{(1a)}$  and  $[A|B]_2^{(3)}$  the row sums of  $A$  are equal to 1, and thus according to (21) we cannot change the inhomogeneous part, except by an overall multiplication. For  $[A|B]_2^{(1b)}$  the gauged inhomogeneous part will turn out to be

$$\begin{bmatrix} (QB)_1 \\ (QB)_2 \end{bmatrix} = \begin{bmatrix} (a-1)(uz - v) \\ (d-1)(uz - v) \end{bmatrix}$$

and by choosing  $v = uz$  we obtain  $(QB)_i = 0$ . For  $[A|B]_2^{(2)}$  we obtain similarly

$$\begin{bmatrix} (QB)_1 \\ (QB)_2 \end{bmatrix} = \begin{bmatrix} -a(uz + v(1-d)) \\ uz + v(1-d) \end{bmatrix}.$$

Now, if  $d \neq 1$  we can again transform to  $(QB)_i = 0$ , but if  $d = 1$  only scaling is possible. Thus, the final form of the solutions of the 2-simplex case is

$$\begin{aligned} [A|B]_2^{(1a)} &= \left[ \begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \end{array} \right] & [A|B]_2^{(1b)} &= \left[ \begin{array}{cc|c} a & 0 & 0 \\ 0 & d & 0 \end{array} \right] & [A|B]_2^{(3)} &= \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right] \\ [A|B]_2^{(2a)} &= \left[ \begin{array}{cc|c} a & 1-a & -az \\ 0 & 1 & z \end{array} \right] & [A|B]_2^{(2b)} &= \left[ \begin{array}{cc|c} a & 1-ad & 0 \\ 0 & d & 0 \end{array} \right] \end{aligned}$$

up to the allowed transformations.

4.2.  $D = 2, 3$ 

For  $D = 2$  the above yields basically two solutions for the 2-simplex equation,  $R$  is either the unit matrix (with possible inhomogeneities) or the permutation matrix  $P$ ,

$$[A|B] = \left[ \begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \end{array} \right] \quad \text{or} \quad \left[ \begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \right] \quad (27)$$

where  $x, y \in \mathbb{Z}_2$ . The same five solutions are obtained by a brute force search without the linearity assumption.

Results (27) work for any  $D$ , but already for  $D = 3$  we obtain other homogeneous solutions, including triangular ones:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

and their reflections.

### 4.3. $D = 4$

The situation is quite different if  $D = 4$ , because of divisors of zero:  $2 \cdot 2 = 0 \pmod{4}$ . In addition to the above generic solutions, we obtain new base solutions

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

and their reflections. When the inhomogeneous parts are added we obtain

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2x \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & y + 2x + gx \\ 2 & 1 & y + gx \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 2x \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 2x \\ 0 & 3 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & 2x \\ 2 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 & y + 2x \\ 2 & 3 & y \end{bmatrix}.$$

In the second solution there is no obvious way to fix the gauge parameter  $g$  and it has been left open.

Thus for 4-state models there seem to be additional symmetries and solutions, and perhaps this case needs to be studied in more detail.

## 5. Results for the 3-simplex equation

Higher simplex equations have many reductions to lower simplex equations, and it is not necessary to repeat them. For example, any solutions of the 2-simplex equation generates a solution of the 3-simplex equation by  $R_{ijk} = R_{ij}\delta_k$  or  $\delta_j R_{jk}$ . These solutions (and those with  $\det A = 0$ ) will not be included in the following list and the solutions below are genuine 3-simplex solutions. Note also that  $R_{ijk} = R_{ik}\delta_j$  (with  $\delta$  on the central index) is not automatically a solution, in particular the permutation matrix  $R_{i_1 i_2 i_3}^{j_1 j_2 j_3} = \delta_{i_1}^{j_3} \delta_{i_2}^{j_2} \delta_{i_3}^{j_1}$  does not solve the tetrahedron equation.

In order to solve the tetrahedron equation under the present ansatz we first consider the homogeneous part. The equation to solve is just like (4) but with  $\mathcal{R}$  replaced by  $A$ . When the matrix

$$A = \begin{bmatrix} a & b & c \\ x & y & z \\ u & v & w \end{bmatrix}$$

is inserted into the  $6 \times 6$  matrix  $\tilde{A}_{K_1 K_2 K_3}$  the six different ways indicated in (4) and we compute the corresponding matrix product we find 29 equations:

$$\begin{aligned} abx = 0 & \quad bxy = 0 & \quad vyz = 0 & \quad vwz = 0 \\ bx(b-x) = 0 & \quad vz(v-z) = 0 & \quad y(bu-cv) = 0 & \quad y(-cx+uz) = 0 \\ b(ay+b-1) = 0 & \quad x(ay+x-1) = 0 & \quad z(wy+z-1) = 0 & \quad v(wy+v-1) = 0 \\ abuz+acx+bcu = 0 & \quad bvxz+cvy+cxy = 0 & \quad buwz+cuw+cvw = 0 \end{aligned}$$



$$\begin{aligned}
abu + acvx + cux = 0 & \quad buy + bvxz + uyz = 0 & \quad cuv + cvwx + uwz = 0 \\
abwz + acz + bcw + c^2 = 0 & \quad auv + avwx + u^2 + uwx = 0 \\
buvz + cuy + cv^2 - cvz = 0 & \quad buxz - bcx + cuy + cx^2 = 0 \\
-b^2u - bcvx + bux - cuy = 0 & \quad -cuy - cvxz + uvz - uz^2 = 0 \\
bwxz + cwy + cxz + cz - c = 0 & \quad abvz + acy + bcv + bc - c = 0 \\
-auy - avxz - uxz - ux + u = 0 & \quad -buv - bvwz - uv - uwy + u = 0 \\
-bcu + bu^2z - c^2vx + cuv + cux - cuz = 0.
\end{aligned}$$

By just considering the first four equations the problem can be split into nine different cases, and each one of them can then be solved rather easily. After eliminating those solutions that reduce to 2-simplex solutions and those with noninvertible  $A$  we find three basic solutions from which others are obtained by the allowed transformations. These solutions and their nonhomogeneous additions will be discussed below.

### 5.1.

The first base solution is

$$A_3^{(1)} = \begin{bmatrix} 0 & 1 & -d \\ 1 & 0 & 1 \\ 0 & 0 & d \end{bmatrix}$$

and when inhomogeneities are added it splits into two:

$$[A|B]_3^{(1a)} = \left[ \begin{array}{ccc|c} 0 & 1 & -1 & x \\ 1 & 0 & 1 & y \\ 0 & 0 & 1 & 0 \end{array} \right] \quad [A|B]_3^{(1b)} = \left[ \begin{array}{ccc|c} 0 & 1 & -d & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & d & 0 \end{array} \right].$$

For  $[A|B]_3^{(1a)}$   $x$  or  $y$  can be still eliminated by a gauge transformation, for  $[A|B]_3^{(1b)}$  the gauge freedom has already been used above.

The transpose of  $A_3^{(1)}$  is not obtained by central reflection and therefore constitutes another solution:

$$[A|B]_3^{(1ta)} = \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & z \end{array} \right] \quad [A|B]_3^{(1tb)} = \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -d & 1 & d & 0 \end{array} \right].$$

These forms cannot be changed by gauge, except by  $z \rightarrow uz$ .

### 5.2.

There are two upper triangular solutions

$$A_3^{(2)} = \begin{bmatrix} a & 1-ab & a(bc-1) \\ 0 & b & 1-bc \\ 0 & 0 & c \end{bmatrix} \quad A_3^{(2t)} = \begin{bmatrix} a & 1-ab & c(ba-1) \\ 0 & b & 1-bc \\ 0 & 0 & c \end{bmatrix}.$$

They differ only in the upper right-hand entry and are related by transposition and central reflection (followed by  $a \leftrightarrow c$ ). But since transposition is not a symmetry of the inhomogeneous part they have to be analysed separately.

Depending on which parameters have unit value we obtain three solutions:

$$[A|B]_3^{(2a)} = \left[ \begin{array}{ccc|c} 1 & 1-b & b-1 & x \\ 0 & b & 1-b & -bz \\ 0 & 0 & 1 & z \end{array} \right]$$

$$[A|B]_3^{(2b)} = \left[ \begin{array}{ccc|c} a & 1-b & a(b-1) & abz \\ 0 & b & 1-b & -bz \\ 0 & 0 & 1 & z \end{array} \right]$$

$$[A|B]_3^{(2c)} = \left[ \begin{array}{ccc|c} a & 1-ab & a(bc-1) & 0 \\ 0 & b & 1-bc & 0 \\ 0 & 0 & c & 0 \end{array} \right].$$

This solution illustrates well how added freedom in  $A$  decreases freedom in  $B$ .

For the transpose  $A_3^{(2t)}$  we obtain two solutions

$$[A|B]_3^{(2ta)} = \left[ \begin{array}{ccc|c} a & 1-a & a-1 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$[A|B]_3^{(2tb)} = \left[ \begin{array}{ccc|c} a & 1-ab & c(ba-1) & 0 \\ 0 & b & 1-bc & 0 \\ 0 & 0 & c & 0 \end{array} \right].$$

5.3.

For the next solution the inhomogeneous terms can always be gauged away and we have

$$[A|B]_3^{(3)} = \left[ \begin{array}{ccc|c} a & 0 & 0 & 0 \\ 1-ab & b & 1-bc & 0 \\ 0 & 0 & c & 0 \end{array} \right].$$

Here we have assumed that at least one of  $a, b, c$  is  $\neq 1$ , else we obtain a diagonal solution with arbitrary  $B$ .

The transpose is again a separate case, we obtain first

$$[A|B]_3^{(3t)} = \left[ \begin{array}{ccc|c} a & 1-ab & 0 & -ay \\ 0 & b & 0 & y \\ 0 & 1-bc & c & -cy \end{array} \right].$$

Now if  $b \neq 1$ , the inhomogeneous part can be eliminated, and we finally have two solutions

$$[A|B]_3^{(3ta)} = \left[ \begin{array}{ccc|c} a & 1-a & 0 & -ay \\ 0 & 1 & 0 & y \\ 0 & 1-c & c & -cy \end{array} \right] \quad [A|B]_3^{(3tb)} = \left[ \begin{array}{ccc|c} a & 1-ab & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 1-bc & c & 0 \end{array} \right].$$

Note how this case is built up from 2-simplex solutions  $[A|B]^{(2b)}$ , but not as simple tensor products.

5.4.  $D = 2$

When all indices are modulo 2 only two solutions remain (in addition to reducible ones), namely

$$\left[ \begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & y \\ 0 & 0 & 1 & 0 \end{array} \right] \quad \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & y \end{array} \right]$$

where  $y \in \mathbb{Z}_2$  and we have used the gauge freedom to eliminate  $B_1$  in the first case. Here it might be useful to record the corresponding  $R$ -matrices for  $y = 0$ :

$$\left( \begin{array}{cccc|cccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{array} \right) \quad \left( \begin{array}{cccc|cccc} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{array} \right).$$

These bear some resemblance with known solutions [5–8].

## 6. Results for the 4-simplex equation

For the 4-simplex case, the  $4 \times 4$  index matrix is embedded into  $10 \times 10$  matrices in four ways. The equations resulting from (5) were solved using the Groebner package of REDUCE [13]. From the results we eliminated those solutions for which  $A$  was in a block form corresponding to  $R$ 's with tensor products form  $R_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} = \delta_{i_1}^{a j_1} M_{i_2 i_3 i_4}^{j_2 j_3 j_4}$ ,  $R_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} = M_{i_1 i_2 i_3}^{j_1 j_2 j_3} \delta_{i_4}^{a j_4}$  or  $R_{i_1 i_2 i_3 i_4}^{j_1 j_2 j_3 j_4} = K_{i_1 i_2}^{j_1 j_2} L_{i_3 i_4}^{j_3 j_4}$  where  $M$  is a solution of the 3-simplex equation and  $K, L$  of the 2-simplex equation. From the remaining list we eliminated all cases obtained from the basic ones by central reflection or by inverse, and those with singular  $A$ . Furthermore, we considered only the generic case of a prime  $D$ .

The solutions  $A$  of the homogeneous equation (and their transposes) were next used as starting points for constructing the nonhomogeneous part  $B$ . Then the continuous gauge freedom was applied to eliminate some freedom from  $B$ . The final result is as follows:

### Permutation blocks

#### 6.1.

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & -1 & b_1 \\ 1 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right].$$

After a gauge transformation (21) we would obtain  $B^t = [b_1 + v, b_2 - v, 0, 0]$  and we could eliminate either  $b_1$  or  $b_2$ .

#### 6.2.

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -a & 1 & a & 1 - ab & 0 \\ 0 & 0 & 0 & b & 0 \end{array} \right].$$

In the generic case we obtain  $B^t = [0, 0, z(b - 1), -z(ab - 1)]$  but this can be eliminated by the gauge transformation. Only if  $a = b = 1$  would we obtain something that cannot be gauged away, but in that case the system reduces to a 3-simplex solution.

6.3.

The transpose of the above solutions is a separate case, and yields

$$\left[ \begin{array}{cccc|c} 0 & 1 & -1 & 0 & b_1 \\ 1 & 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-b & b & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & -a & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 1-ab & b & 0 \end{array} \right]$$

(where  $b_1$  or  $b_2$  could be gauged away). Again when the  $A$  part is restricted, the  $B$  part gains some freedom.

6.4.

The next cases are somewhat similar to the above, we obtain

$$\left[ \begin{array}{cccc|c} 0 & 1 & -a & a-1 & b_1 \\ 1 & 0 & 1 & 0 & b_2 \\ 0 & 0 & a & 1-a & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & -a & ab-1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1-ab & 0 \\ 0 & 0 & 0 & b & 0 \end{array} \right].$$

6.5.

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -a & 1 & a & 0 & 0 \\ a-1 & 0 & 1-a & 1 & x \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -a & 1 & a & 0 & 0 \\ ab-1 & 0 & 1-ab & b & 0 \end{array} \right].$$

6.6.

$$\left[ \begin{array}{cccc|c} 0 & 1 & -a & a & ax \\ 1 & 0 & 1 & -1 & -x \\ 0 & 0 & a & 1-a & -ax \\ 0 & 0 & 0 & 1 & x \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & -a & ab & 0 \\ 1 & 0 & 1 & -b & 0 \\ 0 & 0 & a & 1-ab & 0 \\ 0 & 0 & 0 & b & 0 \end{array} \right].$$

6.7.

For the transpose of the above nothing can be gauged away, and we obtain

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & x \\ 1 & -1 & 0 & 1 & y \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & -bx \\ b & -b & 1-b & b & x \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -a & 1 & a & 0 & 0 \\ a & -1 & 1-a & 1 & x \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -a & 1 & a & 0 & 0 \\ ab & -b & 1-ab & b & 0 \end{array} \right].$$

6.8.

The next  $A$  matrix is invariant under central reflection, and gauge transformation changes nothing. We obtain three different cases

$$\left[ \begin{array}{cccc|c} 1 & 1 & -1 & 0 & x \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & y \end{array} \right] \quad \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -d & 1 & d & y \end{array} \right] \quad \left[ \begin{array}{cccc|c} a & 1 & -a & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -d & 1 & d & 0 \end{array} \right].$$

6.9.

For the transpose of the above the inhomogeneous part is quite different and we obtain

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 1 & 0 & 1 & -1 & y \\ -1 & 1 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 & -y \end{array} \right] \quad \left[ \begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -d & 0 \\ -a & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

In both cases there are two free parameters.

6.10.

$$\left[ \begin{array}{cccc|c} a & 1 & -a & a & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & y \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} a & 1 & -a & ad & 0 \\ 0 & 0 & 1 & -d & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

*Triangular blocks*

6.11.

$$\left[ \begin{array}{cccc|c} 1 & 1-b & b-1 & 1-b & 0 \\ 0 & b & 1-b & b-1 & 0 \\ 0 & 0 & 1 & 0 & x \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} a & 1-ab & c(ab-1) & cd(1-ab) & 0 \\ 0 & b & 1-bc & d(bc-1) & 0 \\ 0 & 0 & c & 1-cd & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.12.

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 1-b & b & 0 & 0 & -bx \\ b-1 & 1-b & 1 & 0 & y \\ 1-b & b-1 & 0 & 1 & z \end{array} \right] \quad \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 1-b & b & 0 & 0 & -bx \\ b-1 & 1-b & 1 & 0 & y \\ d(1-b) & d(b-1) & 1-d & d & -dy \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 1-b & b & 0 & 0 & -xb \\ c(b-1) & 1-bc & c & 0 & cbx \\ c(1-b) & (bc-1) & 1-c & 1 & y \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ 1-ab & b & 0 & 0 & 0 \\ c(ab-1) & 1-bc & c & 0 & 0 \\ cd(1-ab) & d(bc-1) & 1-cd & d & 0 \end{array} \right].$$

6.13.

A detailed analysis of this case leads to some subcases that are identical to those of central reflected (section 6.2.2) and are not repeated here.

$$\left[ \begin{array}{cccc|c} a & 1-ab & a(b-1) & a(1-b) & 0 \\ 0 & b & 1-b & (b-1) & 0 \\ 0 & 0 & 1 & 0 & y \\ 0 & 0 & 0 & 1 & 0 \end{array} \right] \left[ \begin{array}{cccc|c} a & 1-ab & a(bc-1) & ad(1-bc) & 0 \\ 0 & b & 1-bc & d(bc-1) & 0 \\ 0 & 0 & c & 1-cd & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.14.

$$\left[ \begin{array}{cccc|c} a & 1-ab & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 1-bc & c & 1-cd & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.15.

$$\left[ \begin{array}{cccc|c} 1 & 1-b & b-1 & 0 & x \\ 0 & b & 1-b & 0 & -bz \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 1-d & d & -dz \end{array} \right] \left[ \begin{array}{cccc|c} a & 1-a & a-1 & 0 & 0 \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1-d & d & 0 \end{array} \right] \left[ \begin{array}{cccc|c} a & 1-ab & c(ab-1) & 0 & 0 \\ 0 & b & 1-bc & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 1-cd & d & 0 \end{array} \right].$$

6.16.

$$\left[ \begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ 1-ab & b & 0 & 0 & 0 \\ c(ab-1) & 1-bc & c & 1-cd & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.17.

$$\left[ \begin{array}{cccc|c} a & 1-ab & a(b-1) & 0 & abx \\ 0 & b & 1-b & 0 & -bx \\ 0 & 0 & 1 & 0 & x \\ 0 & 0 & 1-d & d & -dx \end{array} \right] \left[ \begin{array}{cccc|c} a & 1-ab & a(bc-1) & 0 & 0 \\ 0 & b & 1-bc & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 1-cd & d & 0 \end{array} \right].$$

6.18.

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 1-b & b & 0 & 0 & -xb \\ bc-1 & 1-bc & c & 1-c & -x(1-bc)-cy \\ 0 & 0 & 0 & 1 & y \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ 1-ab & b & 0 & 0 & 0 \\ a(bc-1) & 1-bc & c & 1-cd & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.19.

Here and in the following case we have rational entries in the index matrix.

$$\left[ \begin{array}{cccc|c} a & 1-a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & x \\ 0 & d-1 & 1-d & d & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} a & 1-ab & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 1/b & 0 & 0 \\ 0 & d-b & 1-d/b & d & 0 \end{array} \right].$$

6.20.

$$\left[ \begin{array}{cccc|c} a & 0 & 0 & 0 & 0 \\ 1-ab & b & 0 & d-b & 0 \\ 0 & 0 & 1/b & (b-d)/b & 0 \\ 0 & 0 & 0 & d & 0 \end{array} \right].$$

6.21.  $D = 2$ .

For  $D = 2$  we have the following new solutions

$$\left[ \begin{array}{cccc|c} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array} \right] \quad \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right].$$

## 7. Discussion

In presenting this (complete) set of linear permutation-type solutions for the 2-, 3- and 4-simplex equations we hope that some of them could be used in other studies. These applications may require further conditions, but we believe that permutation-type solutions are so benign that they should satisfy these conditions, if just independence of any spectral parameter is acceptable.

Another hope is that the solutions can teach us something about the equations themselves. One observation in that direction is that some of the solutions fall into patterns

that seem to continue to any  $n$ . For example,  $A_2^{(2)}$ ,  $A_3^{(3r)}$  and section 6.2.4 start a pattern that seems to continue as

$$\begin{bmatrix} a_1 & 1 - a_1 a_2 & 0 & 0 & 0 & \dots \\ 0 & a_2 & 0 & 0 & 0 & \dots \\ 0 & 1 - a_2 a_3 & a_3 & 1 - a_3 a_4 & 0 & \dots \\ 0 & 0 & 0 & a_4 & 0 & \dots \\ 0 & 0 & 0 & 1 - a_4 a_5 & a_5 & \ddots \\ \vdots & \vdots & \vdots & & \ddots & \ddots \end{bmatrix}.$$

This band structure could make sense even as an infinite matrix, and perhaps we should soon start to think what kind of object the ‘ $\infty$ -simplex’ equation might be.

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### References

- [1] Baxter R J 1981 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
- Jimbo M (ed) 1990 *Yang–Baxter Equations in Integrable Systems* (Singapore: World Scientific)
- [2] Zamolodchikov A B 1980 *Zh. Eksp. Teor. Fiz.* **79** 641–64 (Engl. transl. 1980 *Sov. Phys.–JETP* **52** 325–36)
- Bazhanov V V and Stroganov Yu G 1982 *Teor. Mat. Fiz.* **52** 105–13 (Engl. transl. 1982 *Theor. Mater. Phys.* **52** 685–91)
- Maillet J M and Nijhoff F W 1989 Multidimensional integrable lattices, quantum groups and the  $D$ -simplex equations *Preprint* CERN-TH5595/89, Clarkson University INS #131
- Bazhanov V V and Baxter R J 1992 New solvable lattice models in three dimensions *J. Stat. Phys.* **69** 453–85
- [3] Kulish P P and Sklyanin E K 1982 Quantum spectral transform method. Recent developments *Integrable Quantum Field Theories* ed J Hietarinta and C Montonen (Berlin: Springer)
- [4] Hietarinta J 1992 All solutions to the constant quantum Yang–Baxter equation in two dimensions *Phys. Lett. A* **165** 245–51
- [5] Korepanov I G 1993 Tetrahedral Zamolodchikov algebras corresponding to Baxter’s  $L$ -operator *Commun. Math. Phys.* **154** 85–97
- [6] Hietarinta J 1994 Labelling schemes for tetrahedron equations and dualities between them *J. Phys. A: Math. Gen.* **27** 5727–48
- [7] Sergeev S M, Mangazeev V V and Stroganov Yu G 1995 The vertex formulation of the Bazhanov–Baxter model *Preprint* hep-th/9504035
- [8] Hietarinta J 1993 Some constant solutions to Zamolodchikov’s tetrahedron equations *J. Phys. A: Math. Gen.* **26** L9–15
- [9] Kashaev R M and Sergeev S M 1996 On pentagon, ten-term, and tetrahedron relations *Preprint* q-alg/9607032
- [10] Frenkel I and Moore G 1991 *Commun. Math. Phys.* **138** 259
- [11] Carter J S and Saito M 1996 On formulation and solutions of simplex equations *Int. J. Mod. Phys. A* **11** 4453–63
- [12] Korepanov I G 1995 Algebraic integrable dynamic systems,  $2 + 1$  dimensional models in wholly discrete space-time, and inhomogeneous models in 2-dimensional statistical physics *Preprint* solv-int/9506003
- [13] Melenk H and Neun W 1995 *GROEBNER: A Package for Calculating Gröbner Bases*, available with REDUCE 3.6 ©1995 RAND.